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A Relativistic Field Theory of the Electron

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Abstract

A set of nonlinear partial differential equations covariant in a non-Euclidean space is reduced to the Dirac equation for the electron and the Maxwell–Lorentz equations of electromagnetic fields under certain assumptions. In the course of reduction, we have opportunities for understanding the relationship between the Dirac equation and the Maxwell–Lorentz equations, and also for visualizing conditions which limit feasible applications of those known equations in physics.

1. *Introduction*

In a paper published earlier (Koga, 1975c), it is suggested that the electron is a localized and self-sustained field governed by a set of nonlinear partial differential equations covariant in a non-Euclidean sense, and that it is possible to reduce those fundamental equations, by linearization, to the Dirac equation for the electron and the Maxwell–Lorentz equations of electromagnetic fields, under two sets of restrictive conditions, respectively. The purpose of the present paper is to demonstrate the feasibility of the suggestion. For the time being, we do not intend to compare solutions of the fundamental equations directly with empirical information. Instead, we take the point of view that the feasibility of those equations is substantiated by the fact that they are reducible to the Dirac equation and the Maxwell–Lorentz equations, of which physical implications are known. Therefore, we concern ourselves with the consistency and compatibility among those conditions under which the reductions are carried out. We expect also that the present investigation will shed some light on those mazing difficulties which we encounter when we try to comprehend the behavior of the electron according to the Dirac equation and the Maxwell–Lorentz equations (Koga, 1975a, 1975b, and 1975c).

First, in Section 2, we shall propose a set of equations as governing the field constituting an electron. In Section 3, we shall show that the original set of equations is reducible, under certain assumptions, to the Dirac equation for a *free* electron. In Section 4, the interaction between an electron and an external electromagnetic field, as represented in the Dirac equation, will be investigated. The derivation of the Maxwell-Lorentz equations will be treated in Section 5. Finally, in Section 6, we shall examine the consistency and compatibility among those assumptions and conditions under which the derivations are made.

2. *Fundamental Equations*

Following Einstein (Einstein, 1919), we assume that an electron consists of two parts: One part is a matter field and the other a gravitation field. The two fields interact mutually and make the electron localized and self-sustained.[1]

We write for the matter field

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{ij})}{\partial x^j} - g^{ij}\frac{\partial \eta}{\partial x^j} = 0 \tag{2.1}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}F^{*ij})}{\partial x^j} + g^{ij}\frac{\partial \xi}{\partial x^j} = 0 \tag{2.2}$$

In the above, g is the determinant of the metric tensor g_{ij}; F^{ij} is an antisymmetric tensor and F^{*ij} is conjugate to F^{ij}; ξ and η are scalars.[2] The equations are covariant in the Riemannian sense (Møller, 1952, p. 283). Since we expect that those equations will be reduced to the Dirac equation, as represented in Appendix A, and also to the Maxwell-Lorentz equations, we write for F^{ij}

$$F^{ij} = \begin{pmatrix} 0 & Q_z & -Q_y & -P_x \\ -Q_z & 0 & Q_x & -P_y \\ Q_y & -Q_x & 0 & -P_z \\ P_x & P_y & P_z & 0 \end{pmatrix} \tag{2.3}$$

Considering

$$\begin{aligned} F^{*ij} &= g^{ik}g^{jm}F^*_{km} \\ &= g^{ik}g^{jm}\tfrac{1}{2}\sqrt{g}\,\delta_{kmst}F^{st} \end{aligned} \tag{2.4}$$

where δ_{kmst} is the Levi-Civita symbol, we have

$$F^*_{ij} = \begin{pmatrix} 0 & -P_z & P_y & Q_x \\ P_z & 0 & -P_x & Q_y \\ -P_y & P_x & 0 & Q_z \\ -Q_x & -Q_y & -Q_z & 0 \end{pmatrix} \times \sqrt{-g}$$

[1] Generally speaking, this idea was proposed by Einstein. However, perhaps due to his esteem of Ernst Mach, Einstein did not necessarily seem to think that the field can be completely self-sustained. This matter has been discussed by some authors. See essays by O. Klein (1971) and H. Margenau (Schilpp, 1949, p. 243).

[2] Most of the mathematical symbols appearing in this paper are, unless otherwise specified, similar to those in (Møller, 1952).

$$F^{*ij} = \begin{pmatrix} 0 & -P'_z & P'_y & -Q'_x \\ P'_z & 0 & -P'_x & -Q'_y \\ -P'_y & P'_x & 0 & -Q'_z \\ Q'_x & Q'_y & Q'_z & 0 \end{pmatrix} \tag{2.5}$$

We note that

$$\mathbf{P} \neq \mathbf{P}', \qquad \mathbf{Q} \neq \mathbf{Q}' \tag{2.6}$$

in general.[3]

In order to see what is necessary for reducing equations (2.1) and (2.2), we rewrite them in terms of **P**, **Q**, **P**′, and **Q**′ given by (2.3) and (2.5):

$$\operatorname{curl} \mathbf{Q} - \frac{\partial \mathbf{P}}{\partial(ct)} - \operatorname{grad} \eta + \frac{1}{\sqrt{-g}}\left[(\operatorname{grad} \sqrt{-g}) \times \mathbf{Q} - \frac{\partial\sqrt{-g}}{\partial(ct)}\mathbf{P}\right] + \mathbf{H} = 0 \tag{2.7}$$

$$\operatorname{div} \mathbf{P} + \frac{\partial \eta}{\partial(ct)} + \frac{1}{\sqrt{-g}}(\operatorname{grad}\sqrt{-g}) \cdot \mathbf{P} + \mathrm{H}_t = 0 \tag{2.8}$$

$$\operatorname{curl} \mathbf{P}' + \frac{\partial \mathbf{Q}'}{\partial(ct)} - \operatorname{grad} \xi + \frac{1}{\sqrt{-g}}\left[(\operatorname{grad} \sqrt{-g}) \times \mathbf{P}' + \frac{\partial\sqrt{-g}}{\partial(ct)}\mathbf{Q}'\right] + \mathbf{\Xi} = 0 \tag{2.9}$$

$$\operatorname{div} \mathbf{Q}' - \frac{\partial \xi}{\partial(ct)} + \frac{1}{\sqrt{-g}}(\operatorname{grad} \sqrt{-g}) \cdot \mathbf{Q}' + \Xi_t = 0 \tag{2.10}$$

where

$$(x, y, z, ct) = (x^1, x^2, x^3, x^4)$$

$$\mathbf{H} = (\mathrm{H}_x, \mathrm{H}_y, \mathrm{H}_z)$$

$$\mathbf{\Xi} = (\Xi_x, \Xi_y, \Xi_z)$$

and

$$\begin{aligned}
\mathrm{H}_x &= \left[-(g^{11}-1)\frac{\partial}{\partial x} - g^{12}\frac{\partial}{\partial y} - g^{13}\frac{\partial}{\partial z} - g^{14}\frac{\partial}{\partial(ct)}\right]\eta \\
\mathrm{H}_y &= \left[-g^{21}\frac{\partial}{\partial x} - (g^{22}-1)\frac{\partial}{\partial y} - g^{23}\frac{\partial}{\partial z} - g^{24}\frac{\partial}{\partial(ct)}\right]\eta \\
\mathrm{H}_z &= \left[-g^{31}\frac{\partial}{\partial x} - g^{32}\frac{\partial}{\partial y} - (g^{33}-1)\frac{\partial}{\partial z} - g^{34}\frac{\partial}{\partial(ct)}\right]\eta \\
\mathrm{H}_t &= \left[-g^{41}\frac{\partial}{\partial x} - g^{42}\frac{\partial}{\partial y} - g^{43}\frac{\partial}{\partial z} - (g^{44}+1)\frac{\partial}{\partial(ct)}\right]\eta
\end{aligned} \tag{2.11}$$

[3] We shall often write **P** for (P_x, P_y, P_z) and **Q** for (Q_x, Q_y, Q_z) simply for the sake of convenience. But they are not three-vectors.

$$\boldsymbol{\Xi} = -\mathbf{H} \text{ where } \eta \text{ is replaced with } -\xi$$
$$\Xi_t = \mathrm{H}_t \text{ where } \eta \text{ is replaced with } -\xi \tag{2.12}$$

By comparing equations (2.7)–(2.10) with equations (A5)–(A8) derived from the Dirac equation in Appendix A, we see easily that those two sets of equations are much similar, if we write

$$\mathbf{P} = \mathbf{P}', \qquad \mathbf{Q} = \mathbf{Q}' \tag{2.13}$$

We expect that the equivalency between the two sets of equations, to a good approximation, will be seen, if we have a proper set of conditions as regards the metric tensor. With this expectation, we wish to consider the well-known Einstein equation for the curvature tensor:

$$R_{ij} - \tfrac{1}{2}g_{ij}R = -\kappa T_{ij} \tag{2.14}$$

where R_{ij} is the contracted curvature tensor, R is the curvature scalar, and T_{ij} is the energy-momentum tensor of the matter field. Einstein gave this equation by considering that the only fundamental tensors that do not contain derivatives of g_{ij} beyond the second order are functions of g_{ij} and the Riemann-Christoffel curvature tensor, and that the equation is analogous to the Poisson equation for the gravitation field of the nonrelativistic limit (Schilpp, 1949, p. 73; Eddington, 1924, p. 79). However, we see two great difficulties in relying on equation (2.14). In the first place, we do not know exactly what T_{ij} is if it is assumed to be a function of variables of the matter field.[4] Secondly, it is an extremely difficult task to treat ten simultaneous partial differential equations of the second order. Our present purpose is to show that the Dirac equation and the Maxwell–Lorentz equations, which are covariant only in the Euclidean sense, are both attainable by linearization of the same one set of nonlinear equations covariant in a non-Euclidean sense. In view of this, we consider that it may not be necessary for the covariancy of the original equations to be so general as to be Riemannian.[5]

In general, if the linearization is made by replacing a scalar function with a scalar constant, a vector that is a function with a vector that is a constant, and so on, the covariancy of the resultant equations is the same as that of the original equations. Furthermore, the same linearization may be made in any coordinate system. In view of the fact that neither the Dirac equation nor the Maxwell–Lorentz equations are covariant in any non-Euclidean sense, we can infer that the linearization in question by which those equations are obtained is to be made by substituting constants, such as the rest mass and the charge

[4] Authors often place emphasis on Hamilton's principle of variation of deriving covariant equations from a Lagrangian function. But the choice of the Lagrangian function is rather arbitrary, and so are variation methods. There is no assurance of uniqueness of the result. As Eddington remarked earlier (1924, p. 139), the physical significance of the methods is unknown and doubtful, particularly when we have no means of evaluating those resultant equations directly in comparison with empirical information.

[5] Because of the restrictive condition, viz., equation (2.14), Einstein's geometry is less general than that of Riemann (Eddington, 1924, p. 82).

of the electron, for functions that are not scalars in the original equations considered in particular coordinate systems. Hence, it is essential, prior to linearization, to make feasible assumptions with respect to the characteristics of the fields and also to choose proper coordinate systems in which the process of linearization is carried out. We consider the following two assumptions:

1. *Assumption as Regards the Gravitation Field.* We assume that the field in question is spatially localized. In the space outside the field, the Riemann-Christoffel tensor is negligibly small.[6] Hence, it is convenient to take a Cartesian coordinate system and represent the metric tensor by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.15}$$

The ten components of the metric tensor that is symmetric are functions of the coordinates when the coordinates are extended within the field of the electron.[7] The ten functions are dependent not only on the manner of extending the coordinates within the field but also on the characteristics of the gravitation field that is induced by the matter field. From the mathematical point of view, in general, it is possible to choose a coordinate system in which four, at most, of the ten components of the metric tensor are specified.[8] Besides, owing to particular characteristics of the matter field, the gravitation field must also be particular. Indeed, we, being positioned outside the field, notice not only that the field is localized in space, but also that there is a certain anisotropy of the field as related to the spin of the electron. We have chosen the coordinates outside the field so that the metric tensor is given by (2.15). We now assume that it is possible to extend the coordinates within the field so that the metric tensor is given by

$$g_{ij} = \begin{pmatrix} g_{11} & 0 & 0 & 0 \\ 0 & g_{22} & 0 & 0 \\ 0 & 0 & g_{33} & g_{34} \\ 0 & 0 & g_{34} & g_{44} \end{pmatrix} \tag{2.16}$$

[6] It would be more reasonable to regard a part of the space as outside when the Riemann-Christoffel tensor is negligibly small in that part. Also we note that, owing to the other bodies of matter contained in the universe, the tensor in question does not completely vanish at any point of the space. But our interest is in the local field, i.e., the electron. Hence, we ignore the curvature of the global scale, and consider an inertial frame of reference.

[7] It is difficult to bring in any measuring rod and clock within the electron. Hence the procedure in question is merely mathematical rather than operational. This matter will be discussed again in Section 6.

[8] This is due to the fact that a four-dimensional continuum obeying Riemannian geometry can be represented graphically as a surface of four dimensions drawn in an Euclidean hyperspace of ten dimensions (Eddington, 1924, Section 65).

to a good approximation. This assumption implies that the directional peculiarity of the field arises always along the extension of the third coordinate. An observer may happen to see that the third coordinate is particular with respect to an electron at a moment of time. But it is natural for the observer to expect that such a direction may change as time passes. The change may occur as a causal event. Nevertheless, we assume that the same direction continues to be particular steadfastly. Indeed, we shall see later on that this condition given rather artificially is related to a particular specification of the spin matrices contained in the Dirac equation, viz., specifying the z component to be diagonal. This obvious correlation supports our previous observation that the Dirac equation has no agency of governing or motivating the temporal change of the anisotropic structure of the wave function that satisfies the Dirac equation (Koga, 1975a). Furthermore, according to another investigation made elsewhere (Koga, 1975c), the z direction seems to be the direction of the spin magnetic moment. If we assume that the difference between the metric tensor given by (2.16) and the one given by (2.15) is small, so that

$$g_{11} \doteqdot g_{22} \doteqdot g_{33} \doteqdot -g_{44} \doteqdot 1 \qquad |g_{34}| < 1 \tag{2.17}$$

then we have

$$g^{11} \doteqdot \frac{1}{g_{11}}, \quad g^{22} \doteqdot \frac{1}{g_{22}}, \quad g^{33} \doteqdot \frac{1}{g_{33}}, \quad g^{44} \doteqdot \frac{1}{g_{44}}$$
$$g^{34} \doteqdot \frac{-g_{34}}{g_{33}g_{44}}, \quad g^{12} = g^{13} = g^{14} = g^{23} = g^{24} = 0 \tag{2.18}$$

according to

$$g^{ij} = A^{ij}/g$$

where A^{ij} is the conjugate minor of the component g_{ij} in the determinant of the tensor.

2. *The Equation for the Gravitation Field.* According to the above observation, we see that equation (2.14) is overly general for the present purpose of linearizing equations (2.7)-(2.10). Instead, we assume a set of equations for the gravitation field that does not contain derivatives of g_{ij} beyond the first order:

$$\frac{1}{\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial x^i} = ag_{ij}(F^{*jk} - g^{jk}\xi)\frac{\partial \eta}{\partial x^k} + bg_{ij}(F^{jk} - g^{jk}\eta)A_k \tag{2.19}$$

where A_k represents an external electromagnetic field of a macroscopic scale, and a and b are scalars.[9] As is well known,

[9] One might ask whether

$$a'g_{ij}(F^{jk} - g^{jk}\eta)\,\partial\xi/\partial x^k$$

is equivalent, for our present purpose, to the first term in the right-hand side of equation (2.19). The answer is negative. This situation seems to be due to the fact that equations (A10)-(A13) are not symmetric with respect to **P** and **Q**. Also see Appendix C.

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^i} \equiv 2\Gamma^i_{ij} \tag{2.20}$$

is not a vector in the Riemannian space. It is a vector if the coordinate transformation coefficient $\alpha^i{}_j$ and the inverse transformation coefficient $\breve{\alpha}_j{}^i$ satisfy

$$\alpha_j{}^i\, \partial\breve{\alpha}_k{}^j/\partial x'^i = 0 \tag{2.21}$$

(Mϕller, 1952, pp. 273, 278.) This relation is equivalent to

$$\partial\breve{\alpha}_k{}^i/\partial x^i = 0 \tag{2.21'}$$

and these transformations constitute a group (see Appendix B). Thus we have a geometry which is less general than Riemannian geometry, and is much more general than Euclidean geometry; note that

$$\partial\breve{\alpha}_k{}^i/\partial x^j \neq 0$$

in the former, while

$$\partial\breve{\alpha}_k{}^i/\partial x^j = 0 \tag{2.22}$$

in the latter. Obviously, the number of equations (2.21′) is four, while the number of equations (2.22) is 64. If we wish to have an approximate set of equations for a gravitational field that do not contain derivatives of the metric tensor beyond the first order, we have to be satisfied with those conditions proposed in the above.

To sum up, we assume that equation (2.19), accompanied with conditions (2.16) and (2.17), is able to determine the field of the electron sufficiently precisely.

The second term in the right-hand side of equation (2.19) gives the effect of an external electromagnetic field which may be represented by

$$A_k = (\mathbf{A}, -\phi) \tag{2.23}$$

where $\mathbf{A}$ is the vector potential and ϕ is the scalar potential.

We expect that the first term in the right-hand side of equation (2.19) is responsible for the rest mass of the electron when equation (2.19) is substituted in equations (2.7)–(2.10). Here we note that the role of $\partial\eta/\partial x^k$ is similar to that of A_k in the same equation. If we imagine that A_k is exerted by the electron itself, the dominant component would be the fourth one.[10] Considering this, we assume that the fourth component of $\partial\eta/\partial x^k$ is dominant in comparison with the other three components. For treating nonlinear terms in equations (2.7)–(2.10) for the purpose of linearization, we assume that

$$\partial\eta/\partial x^i \doteqdot (0, 0, 0, \partial\eta/\partial x^4) \tag{2.24}$$

[10] The electron has no velocity relative to itself. This analogy is feasible only in a sense of approximation, for the electron has a spatial extension.

Similarly, we assume that

$$\partial\xi/\partial x^i \doteqdot (0, 0, 0, \partial\xi/\partial x^4) \tag{2.25}$$

According to those conditions considered above, we have for the H's given by (2.11)

$$\begin{aligned} &H_x = H_y = 0, \qquad H_z = -g^{34}\, \partial\eta/\partial(ct) \\ &H_t = -(g^{44} + 1)\, \partial\eta/\partial(ct) \end{aligned} \tag{2.26}$$

and for the Ξ's given by (2.12)

$$\begin{aligned} &\Xi_x = \Xi_y = 0, \qquad \Xi_z = g^{34}\, \partial\xi/\partial(ct) \\ &\Xi_t = (g^{44} + 1)\, \partial\xi/\partial(ct) \end{aligned} \tag{2.27}$$

Also we assume that g_{ij} given by (2.16) is sufficiently close to g_{ij} given by (2.15) that condition (2.13)

$$\mathbf{P} = \mathbf{P}', \qquad \mathbf{Q} = \mathbf{Q}' \tag{2.28}$$

is justified to within the given approximation. Indeed, without this condition, equations (2.9) and (2.10) would be much remote from Eqs. (A7) and (A8) given in Appendix A.

For deriving the Dirac equation in Sections 3 and 4, and also for deriving the Maxwell-Lorentz equations in Sections 5, from equations (2.7)-(2.10) and (2.19), we have to assume further a few conditions. But there is no directly empirical means which provides information serving as justification for those conditions. The justification may be found in the following two facts: The first is that the treatment leads to the Dirac equation and the Maxwell-Lorentz equations, which are known to provide information of natural phenomena; and the second is that those conditions assumed case by case, each somehow in an ad hoc manner. are mutually compatible. We shall discuss this matter in a retrospective way again in Section 6.

3. *The Dirac Equation for a Free Electron*

It is possible to derive the Dirac equation for the electron from equations (2.7)-(2.10) and (2.19), by substituting

$$2mc/\hbar$$

for

$$\frac{g^{34}}{\xi}\frac{\partial\eta}{\partial(ct)} \quad \text{and} \quad -\frac{g^{34}}{\eta}\frac{\partial\xi}{\partial(ct)} \tag{3.1}$$

in nonlinear terms. In this section, we assume that there is no external field:

$$A_k = 0$$

in (2.19). The effect of A_k will be investigated in the next section.

On substituting (2.18), (2.24), and (2.25) in (2.19), we have

$$\frac{\operatorname{grad}\sqrt{-g}}{\sqrt{-g}} = -a\mathbf{Q}\frac{\partial\eta}{\partial(ct)} \tag{3.2}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial(ct)} = -a\xi\frac{\partial\eta}{\partial(ct)} \tag{3.3}$$

In order to make equation (2.7) equivalent to equation (A5), with $\mathbf{f}^{(1)}$ being omitted, it is necessary that

$$(1/\sqrt{-g})\{(\operatorname{grad}\sqrt{-g}) \times \mathbf{Q} - [\partial\sqrt{-g}/\partial(ct)]\mathbf{P}\} + \mathbf{H} = 0 \tag{3.4}$$

By substituting (3.2) and (2.26) in (3.4), we obtain

$$\mathbf{P} = (g^{34}/a\xi)\mathbf{I} \tag{3.5}$$

where

$$\mathbf{I} = (0, 0, 1) \tag{3.6}$$

is defined in order to simplify the presentation, and may be treated as if it were a three-vector. On substituting (3.2), (3.3), and (3.5) in the nonlinear terms in (2.8), and considering (3.1), we have

$$\begin{aligned}\frac{\operatorname{grad}\sqrt{-g}}{\sqrt{-g}} \cdot \mathbf{P} + \mathrm{H}_t &= -\frac{g^{34}}{\xi}\frac{\partial\eta}{\partial(ct)}\left(\mathbf{I}\cdot\mathbf{Q} + \frac{g^{44}+1}{g^{34}}\xi\right)\\ &= -\frac{2mc}{\hbar}\left(\mathbf{I}\cdot\mathbf{Q} + \frac{g^{44}+1}{g^{34}}\xi\right)\end{aligned} \tag{3.7}$$

Hence, if we assume that

$$|(g^{44}+1)/g^{34}| \tag{3.8}$$

is negligibly small as of higher order, equation (2.8) is equivalent to equation (A6) with $f^{(2)}$ being omitted.

In equation (2.9), the nonlinear terms are treated by considering (2.27), (2.28), (3.2), (3.3), (3.5), and finally (3.1), as follows:

$$\begin{aligned}&\frac{1}{\sqrt{-g}}\left[(\operatorname{grad}\sqrt{-g}) \times \mathbf{P} + \frac{\partial\sqrt{-g}}{\partial(ct)}\mathbf{Q}\right] + \mathbf{\Xi}\\ &= -\frac{g^{34}}{\xi}\frac{\partial\eta}{\partial(ct)}\mathbf{Q}\times\mathbf{I} - a\frac{\partial\eta}{\partial(ct)}\xi\mathbf{Q} - g^{34}\frac{\partial\xi}{\partial(ct)}\mathbf{I}\\ &= \frac{2mc}{\hbar}\left(\mathbf{I}\times\mathbf{Q} - \frac{a\xi^2}{g^{34}}\mathbf{Q} + \eta\mathbf{I}\right)\end{aligned} \tag{3.9}$$

If the second term in the result is negligible as of higher order, equation (2.9) is equivalent to (A7) with $\mathbf{f}^{(3)}$ being omitted. Postponing this exposition for a while, we treat the nonlinear terms in equation (2.10) as follows:

$$\frac{\operatorname{grad}\sqrt{-g}}{\sqrt{-g}}\cdot\mathbf{Q}+\Xi_t=-a\frac{\partial\eta}{\partial(ct)}Q^2+(g^{44}+1)\frac{\partial\xi}{\partial(ct)}$$
$$=\frac{2mc}{\hbar}\left(\frac{-a\xi}{g^{34}}Q^2-\frac{g^{44}+1}{g^{34}}\eta\right) \tag{3.10}$$

The last term in the above is immediately negligible according to the assumption made for (3.8). In order to evaluate the nonlinear terms left in (3.9) and (3.10), we consider the following:

The spin matrix component in the z direction is given as diagonal in Appendix A. According to an earlier investigation (Koga, 1975c), we have an expectation that $\mathbf{Q}$, representing the magnetic field, is almost in the z direction, i.e., $(\operatorname{grad}\sqrt{-g})_x=(\operatorname{grad}\sqrt{-g})_y=0$, according to (3.2). Hence, considering (3.5), we assume that

$$\mathbf{Q}=-\mu\mathbf{P} \tag{3.11}$$

We shall see that this assumption is consistently necessary also in Sections 4 and 5. Substituting $\mathbf{Q}$ from (3.2) and $\mathbf{P}$ from (3.5) in (3.11), we have

$$\mu=\frac{\xi(\operatorname{grad}\sqrt{-g})_z}{\sqrt{-g}\,g^{34}\partial\eta/\partial(ct)}=\frac{\hbar}{2mc\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial z} \tag{3.12}$$

In Section 5, we shall see that $\mathbf{P}$ represents the electric field in a certain way, and $\mathbf{Q}$ the magnetic field. Considering that the magnetic field is not so significant as the electric field when they are induced by an electron at rest, we assume that

$$|\mu|<1 \tag{3.13}$$

By substituting $\mathbf{P}$ given by (3.5) in (3.11), we have

$$|\mu|=|Q_z/P_z|=|a\xi Q/g^{34}|<1 \tag{3.14}$$

Accordingly, those nonlinear terms left in (3.9) and (3.10) should be neglected. In this way, it has been shown that equations (2.9) and (2.10) are equivalent to equations (A7) and (A8), respectively, if $\mathbf{f}^{(3)}$ and $f^{(4)}$ are omitted from the latter two. In the next section, we shall show that the second term in the right-hand side of equation (2.19) is responsible for the f's that have been omitted thus far in equations (A5)-(A8).

4. *The Dirac Equation for an Electron in an External Electromagnetic Field*

Those terms which represent the effect of an external electromagnetic field on an electron in the Dirac equation yield the f's in equations (A5)-(A8) obtained in Appendix A; the f's are given by (A10)-(A13). In this section, we shall show that those additional terms arise from the second term in the right-hand side of equation (2.19), when $(-g)^{-1/2}\,\partial\sqrt{-g}/\partial x^i$ from (2.19) is substituted in equations (2.1) and (2.2). Once the substitution has been completed, those additional terms are treated on the assumption that the metric tensor is to be given by (2.15). (This seems to imply that the interaction is considered only near the outer edge of the electron field.)

According to (2.3), we write

$$(F^{jk} - g^{jk}\eta)A_k = (\mathbf{S}, S_t) \tag{4.1}$$

where

$$\begin{aligned} \mathbf{S} &= \mathbf{A} \times \mathbf{Q} - \eta\mathbf{A} - A_4\mathbf{P} \\ S_t &= \mathbf{A}\cdot\mathbf{P} + \eta A_4 \\ \mathbf{A} &= (A_x, A_y, A_z) \equiv (A_1, A_2, A_3) \end{aligned} \tag{4.2}$$

Considering (2.5) and (2.15), we have similarly

$$(F^{*jk} + g^{jk}\xi)A_k = (\mathbf{T}, T_t) \tag{4.3}$$

where

$$\begin{aligned} \mathbf{T} &= -\mathbf{A} \times \mathbf{P} + \xi\mathbf{A} - A_4\mathbf{Q} \\ T_t &= \mathbf{A}\cdot\mathbf{Q} - \xi A_4 \end{aligned} \tag{4.4}$$

As is shown in Appendix D, relation (3.11)

$$\mathbf{Q} = -\mu\mathbf{P}, \qquad |\mu| < 1 \tag{3.11}$$

leads to

$$\begin{aligned} \mathbf{S} &= \mu\mathbf{T} \\ S_t &= \mu T_t \end{aligned} \tag{4.5}$$

to the approximation of ignoring μ^2. According to (3.11) and (4.5), we get

$$\begin{aligned} \mathbf{S} \times \mathbf{P} &= -\mathbf{T} \times \mathbf{Q} \\ \mathbf{S}\cdot\mathbf{P} &= -\mathbf{T}\cdot\mathbf{Q} \\ S_t\mathbf{P} &= -T_t\mathbf{Q} \end{aligned} \tag{4.6}$$

By considering (3.5) and (3.11), we have

$$\begin{aligned} \mathbf{Q} &= (\mu g^{34}/a\xi)\mathbf{I} \\ &= (e/b\hbar c)\mathbf{I} \end{aligned} \tag{4.7}$$

where e is defined by

$$e = -b\mu\hbar c g^{34}/(a\xi) \tag{4.8}$$

and b appears in equation (2.19).

We are ready to calculate the f's. We write

$$(1/\sqrt{-g})(\partial\sqrt{-g}/\partial x^i)F^{ki} = (\mathbf{f}^{(1)}, f^{(2)})$$

and substitute

$$(1/\sqrt{-g})(\partial\sqrt{-g}/\partial x^i) = bg_{ij}(F^{jk} - g^{jk}\eta)A_k$$

according to (2.19). Then, considering (4.1), (4.6), (4.7), and (4.2), we have

$$\begin{aligned}\mathbf{f}^{(1)} &= b(\mathbf{S} \times \mathbf{Q} + S_t\mathbf{P})\\ &= b(\mathbf{S} \times \mathbf{Q} - T_t\mathbf{Q})\\ &= b(\mathbf{A} \times \mathbf{Q} - \eta A - A_4\mathbf{P}) \times Q - b(\mathbf{A} \cdot \mathbf{Q} - \xi A_4)\mathbf{Q}\\ &= (e/\hbar c)(\mathbf{A} \times \mathbf{Q} - \eta\mathbf{A} + \phi\mathbf{P}) \times \mathbf{I} - (\mathbf{A} \cdot \mathbf{Q} + \phi\xi)\mathbf{I} \qquad (4.9)\end{aligned}$$

where $A_4 = -\phi$ has been considered. Similarly

$$\begin{aligned}f^{(2)} &= b\mathbf{S} \cdot \mathbf{P}\\ &= -b\mathbf{T} \cdot \mathbf{Q}\\ &= b(\mathbf{A} \times \mathbf{P} - \xi\mathbf{A} + A_t\mathbf{Q}) \cdot \mathbf{Q}\\ &= (e/\hbar c)(\mathbf{A} \times \mathbf{P} - \xi\mathbf{A} - \phi\mathbf{Q}) \cdot \mathbf{I} \qquad (4.10)\end{aligned}$$

Repeating a similar treatment, we obtain $\mathbf{f}^{(3)}$ and $f^{(4)}$, appearing in (A7) and (A8), from

$$\frac{1}{\sqrt{-g}} \frac{\partial\sqrt{-g}}{\partial x^k} F^{*\,ik}$$

in equation (2.2):

$$\begin{aligned}\mathbf{f}^{(3)} &= -b(-\mathbf{S} \times \mathbf{P} + S_t\mathbf{Q})\\ &= (e/\hbar c)(\mathbf{A} \times \mathbf{P} - \xi\mathbf{A} - \phi\mathbf{Q}) \times \mathbf{I} - (\mathbf{A} \cdot \mathbf{P} - \eta\phi)\mathbf{I} \qquad (4.11)\\ f^{(4)} &= b(\mathbf{S} \cdot \mathbf{Q})\\ &= (e/\hbar c)(\mathbf{A} \times \mathbf{Q} - \eta\mathbf{A} + \phi\mathbf{P}) \cdot \mathbf{I} \qquad (4.12)\end{aligned}$$

We note that the sign of $\mathbf{f}^{(3)}$ has been adjusted so as to be consistent with equation (2.9), which is given as corresponding to equation (2.2) where the sign of the left-hand side has been changed.

We note that the f's obtained in the above are exactly the same as those given by (A10)-(A13).

5. *The Maxwell-Lorentz Equations*

According to Lorentz, a superposition of the fields of many electrons, being smeared out in average, is observed as an electromagnetic field governed by the Maxwell-Lorentz equations. In this section, we shall demonstrate that the

Maxwell-Lorentz equations are derivable from equations (2.1), (2.2), and (2.19) assigned to each of many electrons distributed in a space domain. It should be noted that a few assumptions necessary for this derivation have already been given for deriving the Dirac equation in previous sections.

The N electrons under investigation are numbered with ν:

$$\nu = 1, 2, 3, \ldots, N$$

The field of electron ν is represented by

$$F^{ik}(\nu) \equiv F^{ik}(X, \nu), \text{ etc.} \tag{5.1}$$

where X represents independent coordinate variables (x^1, x^2, x^3, x^4). Since there are N electrons, we have to consider that each electron is submerged in the gravitational field of which the metric tensor is affected by the N electrons. The covariant components of the metric tensor are denoted by $\tilde{g}_{ik}$. We write equations (2.1) and (2.2) for each electron:

$$\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^k}(\sqrt{-\tilde{g}}F^{ik}(\nu)) - \tilde{g}^{ik}\frac{\partial \eta(\nu)}{\partial x^k} = 0 \tag{5.2}$$

$$\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial}{\partial x^k}(\sqrt{-\tilde{g}}F^{*ik}(\nu)) + \tilde{g}^{ik}\frac{\partial \xi(\nu)}{\partial x^k} = 0 \tag{5.3}$$

In place of equation (2.19), we have[11]

$$\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^k} = a\tilde{g}_{kj}\sum_{\nu}\,[F^{*jm}(\nu) - \tilde{g}^{jm}\xi(\nu)]\frac{\partial \eta(\nu)}{\partial x^m} \tag{5.4}$$

In order to simplify the treatment, we first assume that there is no significant relative velocity among the electrons so that we may choose coordinate systems in which, as assumed by (2.24),

$$\begin{aligned} \partial\eta(\nu)/\partial x^k &= (0, 0, 0, \partial\eta/\partial x^4) \\ \partial\xi(\nu)/\partial x^k &= (0, 0, 0, \partial\xi/\partial x^4) \end{aligned} \tag{5.5}$$

are valid for all the electrons. (It would be possible to eliminate this assumption, if we intend to tolerate the complicacy of taking a kinetic-theoretical approach.) In addition, for the time being, we assume that those electrons are at rest with respect to the coordinate system. But their spins are directed to various directions at random. Hence, bilinear products of variables with no correlation such as

$$\begin{aligned} &P_x(\nu)\xi(\nu), \quad P_y(\nu)Q_z(\nu), \quad Q_x(\nu)Q_y(\nu), \\ &Q_x(\nu)P_x(\nu'), \quad Q_x(\nu)Q_x(\nu'), \text{ etc.}, \qquad \nu \neq \nu' \end{aligned} \tag{5.6}$$

[11] We assume that the interaction between a pair of electrons does not significantly contribute to $\tilde{g}$.

are assumed to vanish in nonlinear terms, when they are summed up with respect to ν and ν'. On consideration of (5.5), we rewrite (5.4) and obtain

$$\frac{1}{\sqrt{-\tilde{g}}}\,\mathrm{grad}\sqrt{-\tilde{g}} = -a \sum Q(\nu)\frac{\partial\eta(\nu)}{\partial(ct)}$$
$$\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial(ct)} = -a \sum \xi(\nu)\frac{\partial\eta(\nu)}{\partial(ct)} \tag{5.7}$$

Utilizing these, we have for a part of the nonlinear terms of (5.2)

$$\begin{aligned}\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^k}\sum_\nu F^{1k}(\nu) &= \frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^2}\sum Q_z(\nu) - \frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^3}\sum Q_y(\nu)\\ &\quad -\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^4}\sum P_x(\nu)\\ &= -a\sum_\nu\sum_{\nu'} Q_z(\nu)Q_y(\nu')\frac{\partial\eta(\nu')}{\partial(ct)} + a\sum_\nu\sum_{\nu'} Q_y(\nu)Q_z(\nu')\\ &\quad \times\frac{\partial\eta(\nu')}{\partial(ct)} - a\sum\sum P_x(\nu)\xi(\nu')\frac{\partial\eta(\nu')}{\partial(ct)}\\ &= 0\end{aligned} \tag{5.8}$$

We have taken into account that variables given by (5.6) vanish when they are averaged over ν and ν'. Similarly

$$(1/\sqrt{-\tilde{g}})(\partial\sqrt{-\tilde{g}}/\partial x^k)\sum F^{ik}(\nu) = 0, \qquad i = 2, 3 \tag{5.8$'$}$$

On the other hand, by considering (3.5) and (3.1), we have

$$\begin{aligned}\frac{1}{\sqrt{-\tilde{g}}}\frac{\partial\sqrt{-\tilde{g}}}{\partial x^k}\sum F^{4k}(\nu) &= -a\sum\sum\frac{\partial\eta(\nu')}{\partial(ct)}\,\mathbf{Q}(\nu')\cdot\mathbf{P}(\nu)\\ &= -a\sum\frac{\partial\eta(\nu)}{\partial(ct)}\,\mathbf{Q}(\nu)\cdot\mathbf{P}(\nu)\\ &= \sum\frac{\partial\eta(\nu)}{\partial(ct)}\frac{g^{34}(X',\nu)}{\xi}\,\mathbf{I}(\nu)\cdot\mathbf{Q}(\nu)\\ &= -\frac{2mc}{\hbar}\sum Q_s(\nu)\end{aligned} \tag{5.9}$$

In the above treatment, we have considered that $\Sigma\Sigma\mathbf{Q}(\nu')\cdot\mathbf{P}(\nu) = 0$, if $\nu \neq \nu'$, and those of which $\nu = \nu'$, do not vanish; also noted is that $g^{34}(X', \nu)$ implies that each electron is given its own spatial coordinate system such that the third coordinate axis is in the direction of its spin, i.e., of $\mathbf{Q}(\nu)$.[12] Hence $Q_s(\nu)$ is the component of $\mathbf{Q}(\nu)$ in the direction of spin, and is the magnitude of $\mathbf{Q}(\nu)$. We note that $\mathbf{I}(\nu)$ is a unit vector oriented to the direction of spin, according to (3.6).

We define the number density of electrons by[13]

$$n = \sum Q_s(\nu) / \iiint Q_s(\nu)\, dx\, dy\, dz \tag{5.10}$$

According to the empirical information of the Bohr magneton, we put[14]

$$\iiint Q_s(\nu)\, dx\, dy\, dz = \frac{\hbar}{2mc} e \tag{5.11}$$

Hence

$$n = (2mc/e\hbar) \Sigma Q_s(\nu)$$

or

$$\Sigma Q_s(\nu) = ne\hbar/(2mc) \tag{5.12}$$

On substituting (5.12), we have for (5.9)

$$(1/\sqrt{-g})(\partial \sqrt{-g}/\partial x^k) \Sigma F^{4k}(\nu) = -ne \tag{5.13}$$

Similarly, the first term of (5.3) is evaluated:

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^k} \sum_{\nu} F^{*ik}(\nu) = 0, \qquad i = 1, 2, 3 \tag{5.14}$$

$$\frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial x^k} \sum F^{*4k}(\nu) = \mu ne \doteqdot 0 \tag{5.15}$$

where μ has been given by (3.13), and μne is to be ignored in comparison with ne in (5.13).

The other terms in equations (5.2) and (5.3) are rather easily evaluated. We have assumed that the electrons are at rest, and hence the average field should be stationary:

$$\sum \partial F^{i4}/\partial x^4 = \sum \partial F^{*/i4}/\partial x^4 = 0 \tag{5.16}$$

Further, considering (5.5) and that the spins are directed at random among those electrons, we have

$$\sum \tilde{g}^{ik}\, \partial\eta/\partial x^k = \sum \tilde{g}^{ik}\, \partial\xi/\partial x^k = 0 \tag{5.17}$$

[12] It is possible to do so, since $\mathbf{Q} \cdot \mathbf{P}$ is a scalar.

[13] We suppose that $Q_s(\nu)$ is uniform inside a space domain and vanishes outside the domain. Then the volume of the domain, V, is given by

$$V = \iiint Q_s(\nu)\, dx\, dy\, dz / Q_s(\nu)$$

[14] Earlier we defined the same e by (4.7). In fact, however, what is defined by (4.7) is e/b. Comparing (4.7) with (5.11), we obtain for b

$$b = 2mV/\hbar^2$$

where V is defined in the previous footnote.

Summarizing the above, and considering (2.3) and (2.5), we have for (5.2) and (5.3)

$$\begin{aligned} &\operatorname{curl} \tilde{\mathbf{Q}}_0 = 0, \quad \operatorname{div} \tilde{\mathbf{P}}_0 = ne \\ &\operatorname{curl} \tilde{\mathbf{P}}_0 = 0, \quad \operatorname{div} \tilde{\mathbf{Q}}_0 = 0 \end{aligned} \tag{5.18}$$

where

$$\tilde{\mathbf{P}}_0 = \sum \mathbf{P}(\nu), \quad \tilde{\mathbf{Q}}_0 = \sum \mathbf{Q}(\nu) \tag{5.19}$$

Of course, those conditions, such as (3.5), (3.11), etc., which are feasible for evaluating non-linear terms to an approximation, should not be applied to those variables contained in (5.18) and (5.19).

Thus far, the electrons have been considered to be at rest to the coordinate system, and equations (5.18) are the Maxwell–Lorentz equations in the system. They are no longer covariant under the transformation defined by (2.21). We consider a new coordinate system moving with velocity $-\mathbf{U}$ relative to the initial one. We note that $dx\, dy\, dx\, d(ct)$ is a pseudoinvariant in the Euclidean space (Eddington, 1924, Section 49). Hence, n defined by (5.10) is the fourth component of a four-vector, and we have, with respect to the new coordinate system,

$$\rho = ne/(1 - U^2/c^2)^{1/2}, \qquad \mathbf{J} = \rho \mathbf{U} \tag{5.20}$$

(Møller, 1952, p. 197). These represent the charge and current densities, respectively. Thus, equations (5.18) yield

$$\begin{aligned} &\operatorname{curl} \tilde{\mathbf{Q}} - \partial \tilde{\mathbf{P}}/\partial(ct) = \mathbf{J}/c \\ &\operatorname{div} \tilde{\mathbf{P}} = \rho \\ &\operatorname{curl} \tilde{\mathbf{P}} + \partial \tilde{\mathbf{Q}}/\partial(ct) = 0 \\ &\operatorname{div} \tilde{\mathbf{Q}} = 0 \end{aligned} \tag{5.21}$$

due to the Lorentz transformation. We may substitute $\mathbf{E}$ for $\tilde{\mathbf{P}}$, and $\mathbf{H}$ for $\tilde{\mathbf{Q}}$.

6. *Summary and Retrospective Remarks*

The fundamental equations given by (2.1), (2.2), and (2.19) are covariant in coordinate systems in a non-Euclidean space defined by (2.21). The linearization of the fundamental equations, leading to the Dirac equation and/or the Maxwell–Lorentz equations, is carried out by substituting constants that are scalars, such as $m/\hbar$ and $e/\hbar$, in place of functions that are not scalars. Therefore, the resultant equations are no longer covariant under the transformation defined by (2.21). Also the characteristics of the electron field which are embodied in the fundamental equations are partly truncated when they are transferred to the resultant equations, viz., the Dirac equation and/or the Maxwell–Lorentz equations.

There are three main assumptions that play significant roles in the process of linearization.

Assumption 1. The direction in which the component of the spin observable $\boldsymbol{\sigma}$ is diagonal is assumed to be particular also for the matter field and the gravitation field. This correlation is suggested by the following: (1) there is no equation of motion for $\boldsymbol{\sigma}$ which is derivable from the Dirac equation (Koga, 1975a); (2) the direction in question is the direction of the magnetic force that is assumed to be embodied in an electron (Koga, 1975c, Section 4).

Assumption 2. $e/\hbar$ and $m/\hbar$ are both proportional to g^{34} of the metric tensor when the third direction is the one in which the component of the spin observable is assumed to be diagonal and the fourth direction is the time direction. See (3.1) and (4.7). The latter is compatible with (5.11).

Assumption 3. Equation (3.4) is assumed simply for the purpose of making equation (2.7) equivalent to equation (A5). This assumption, together with Assumption 1, leads to relations (3.11) and (3.13)

$$\mathbf{Q} = -\mu \mathbf{P}, \qquad |\mu| < 1$$

These are necessary also in Sections 4 and 5.

We notice that Assumption 1 is very restrictive. For deriving the Maxwell-Lorentz equations, the effect of A_k in (2.19) is assumed simply to be negligible; also assumed is that the interaction between a pair of electrons does not contribute at all to the gravitation field. If those effects are assumed to be significant, the resultant equations will contain nonlinear terms, i.e., we shall get a nonlinear version of the Maxwell-Lorentz equations.

Supposing that equations (2.1), (2.2), and (2.19) are given, is it possible to derive the Dirac equation and/or the Maxwell-Lorentz equations, with no knowledge of those resultant equations prior to the derivation? In order to make it possible to do so, the following two conditions are necessary: (1) The fundamental equations provide us a set of detailed information of the characteristics of the electron; (2) reduction of the fundamental equations to the Dirac equation or the Maxwell-Lorentz equations is a completely logical process, once certain choices of characteristics of the electron to be embodied in those resultant equations have been made. We notice that these two conditions do not exist. In the first place, there is no directly operational means of comparing solutions of those fundamental equations with the fields constituting the electron; we cannot bring in any measuring devices within an electron. The fundamental equations by themselves do not provide any knowledge of an electron that may be compared directly with empirical informations. We have to make assumptions that define the correspondence between empirical informations and those that the fundamental equations may provide. Those assumptions must constitute one consistent set. We recognize that this process of interpretation of the fundamental equations is no more than the process of reducing them to equations, such as the Dirac equation and the Maxwell-Lorentz equations, of which empirical meanings are accessible.

Finally, on considering together the set of fundamental equations (2.1), (2.2), and (2.19), the Dirac equations, Schrödinger's time-dependent wave equation, and Newton's equations of motion of a material point, we recognize that those four sets of equations constitute a hierarchy of laws governing the motion of the electron, in the sense that each is derivable from the one that precedes it in rank (Koga, 1975b). But the Maxwell-Lorentz equations govern a superposition of fields averaged over many electrons, and are to be placed outside the hierarchy. It is unknown, at this moment, if it is possible to place another set of equations between the fundamental equations and the Dirac equation.

Appendix A: A Representation of the Dirac Equation

We write for the Dirac equation for the electron

$$\left[i\hbar \frac{\partial}{\partial(ct)} + \frac{e\phi}{c}\right]\Psi - \boldsymbol{\alpha}\cdot\left(-i\hbar\frac{\partial}{\partial \mathbf{r}} + \frac{e\mathbf{A}}{c}\right)\Psi - \beta mc\Psi = 0 \qquad \text{(A1)}$$

where $e < 0$, $\mathbf{r} = (x, y, z)$, and $(A, i\phi)$ is the four-potential of an electromagnetic field exerted on the electron. By taking

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(A2)}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{(A3)}$$

we write for Ψ

$$\begin{aligned} \Psi_i &= \theta_i \exp(-imc^2 t/\hbar), \qquad i = 1, 2, 3, 4 \\ \theta_1 &= iP_x + P_y \\ \theta_2 &= -iP_z + \xi \\ \theta_3 &= i(iQ_x + Q_y) \\ \theta_4 &= i(-iQ_z + \eta) \end{aligned} \qquad \text{(A4)}$$

and substitute these in (A1), obtaining

$$\text{curl}\,\mathbf{Q} - \partial\mathbf{P}/\partial(ct) - \text{grad}\,\eta + \mathbf{f}^{(1)} = 0 \qquad \text{(A5)}$$

$$\text{div}\,\mathbf{P} + \partial\eta/\partial(ct) - (2mc/\hbar)\mathbf{Q}\cdot\mathbf{I} + f^{(2)} = 0 \qquad \text{(A6)}$$

$$\text{curl}\,\mathbf{P} + \partial\mathbf{Q}/\partial(ct) - \text{grad}\,\xi - (2mc/\hbar)(\mathbf{Q}\times\mathbf{I} - \eta\mathbf{I}) + \mathbf{f}^{(3)} = 0 \qquad \text{(A7)}$$

$$\text{div}\,\mathbf{Q} - \partial\xi/\partial(ct) + f^{(4)} = 0 \qquad \text{(A8)}$$

where

$$\begin{aligned} \mathbf{P} &= (P_x, P_y, P_z) \\ \mathbf{Q} &= (Q_x, Q_y, Q_z) \\ \mathbf{I} &= (0, 0, 1) \end{aligned} \qquad \text{(A9)}$$

and

$$\mathbf{f}^{(1)} = e(\hbar c)^{-1}[(\mathbf{A} \times \mathbf{Q} - \eta\mathbf{A} + \phi\mathbf{P}) \times \mathbf{I} - (\mathbf{A}\cdot\mathbf{Q} + \phi\xi)\mathbf{I}] \quad \text{(A10)}$$

$$f^{(2)} = e(\hbar c)^{-1}(\mathbf{A} \times \mathbf{P} - \xi\mathbf{A} - \phi\mathbf{Q})\cdot\mathbf{I} \quad \text{(A11)}$$

$$\mathbf{f}^{(3)} = e(\hbar c)^{-1}[(\mathbf{A} \times \mathbf{P} - \xi\mathbf{A} - \phi\mathbf{Q}) \times \mathbf{I} - (\mathbf{A}\cdot\mathbf{P} - \phi\eta)\mathbf{I}] \quad \text{(A12)}$$

$$f^{(4)} = e(\hbar c)^{-1}(\mathbf{A} \times \mathbf{Q} - \eta\mathbf{A} + \phi\mathbf{P})\cdot\mathbf{I} \quad \text{(A13)}$$

Appendix B: Transformation Coefficients Satisfying (2.21)

1. The proof of the equivalency between (2.21) and (2.21′) is given as follows:

$$\begin{aligned}\alpha_j{}^i\, \partial\breve{\alpha}_k{}^j/\partial x'^i &= \alpha_j{}^i(\partial\breve{\alpha}_k{}^j/\partial x^m)(\partial x^m/\partial x'^i)\\ &= \alpha_j{}^i\breve{\alpha}_i{}^m\, \partial\breve{\alpha}_k{}^j/\partial x^m\\ &= \delta_j{}^m\, \partial\breve{\alpha}_k{}^j/\partial x^m\\ &= \partial\breve{\alpha}_k{}^i/\partial x^i \end{aligned} \quad \text{(B1)}$$

2. In order to be a group, a set of elements must be subject to the following four conditions of combination: (1) The product of two elements is an element; (2) a unit element exists; (3) the inverse element for an element exists; (4) the associative law is satisfied. It is rather easy to see that they satisfy conditions (2) (3), and (4). In the following, we shall show that condition (1) is satisfied by the transformations under consideration.

We suppose that α denotes the coefficient of transformation from system A to system B, β the coefficient of transformation from system B to system C, and γ the coefficient of transformation from system A to system C. Their inverse coefficients are denoted by $\breve{\alpha}$, $\breve{\beta}$, and $\breve{\gamma}$, respectively. Accordingly,

$$\begin{aligned} dx'^k &= \alpha_i{}^k\, dx^i\\ dx''^k &= \beta_i{}^k\, dx'^i = \beta_m{}^k\alpha_i{}^m\, dx^i = \gamma_i{}^k\, dx^i \end{aligned} \quad \text{(B2)}$$

Hence, we have

$$\gamma_i{}^k = \beta_m{}^k\alpha_i{}^m, \qquad \breve{\gamma}_i{}^k = \breve{\alpha}_m{}^k\breve{\beta}_i{}^m \quad \text{(B3)}$$

By differentiation, we have from (B3)

$$\partial\breve{\gamma}_i{}^k/\partial x^k = (\partial\breve{\alpha}_j{}^k/\partial x^k)\breve{\beta}_i{}^j + \breve{\alpha}_j{}^k\, \partial\breve{\beta}_i{}^j/\partial x^k \quad \text{(B4)}$$

According to (B2), we have the following reduction:

$$\begin{aligned}\breve{\alpha}_j{}^k\, \partial\breve{\beta}_i{}^j/\partial x^k &= \breve{\alpha}_j{}^k(\partial x'^m/\partial x^k)(\partial\breve{\beta}_i{}^j/\partial x'^m)\\ &= \breve{\alpha}_j{}^k\alpha_k{}^m\, \partial\breve{\beta}_i{}^j/\partial x'^m\\ &= \delta_j{}^m\, \partial\breve{\beta}_i{}^j/\partial x'^m\\ &= \partial\breve{\beta}_i{}^m/\partial x'^m \end{aligned} \quad \text{(B5)}$$

Accordingly, (B4) yields

$$\partial\breve{\gamma}_i{}^k/\partial x^k = (\partial\breve{\alpha}_j{}^k/\partial x^k)\breve{\beta}_i{}^j + \partial\breve{\beta}_i{}^m/\partial x'^m \tag{B6}$$

If $\breve{\alpha}$ and $\breve{\beta}$ satisfy condition (2.21), i.e.,

$$\partial\breve{\alpha}_k{}^i/\partial x^i = 0, \qquad \partial\breve{\beta}_i{}^m/\partial x'^m = 0$$

then we have, according to (B6),

$$\frac{\partial\breve{\gamma}_i{}^k}{\partial x^k} = 0$$

In other words, $\breve{\gamma}$ satisfies condition (2.21) also.

Appendix C: A Theorem for Justification of Equation (2.19)

The covariant divergence of a vector a^i is given by

$$\text{div}\ \{a^i\} = \frac{\partial a^i}{\partial x^i} + \frac{a^i}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^i} \tag{C1}$$

(Møller, 1952, p. 283).

We have from (2.1)

$$\text{div}\ \{g^{ik}\ \partial\eta/\partial x^k\} = \text{div}\ \{(1/\sqrt{-g})(\partial/\partial x^k)(\sqrt{-g}\,F^{ik})\}$$

Considering (C1) and noticing that F^{ik} is an antisymmetric tensor, we see that the right-hand side of the above equation vanishes. Hence

$$\text{div}\ \{g^{ik}\ \partial\eta/\partial x^k\} = 0 \tag{C2}$$

Similarly,

$$\text{div}\ \{g^{ik}\ \partial\xi/\partial x^k\} = 0$$

On the other hand, in the Euclidean space, we have the so-called Lorentz relation

$$\partial A^k/\partial x^k = 0 \tag{C3}$$

or

$$\text{div}\, A + \partial\phi/\partial(ct) = 0$$

Comparing (C2) and (C3), we see that the two terms in the right-hand side of equation (2.19) are mutually analogous.

Appendix D: Proof of Relation (4.5)

We substitute (4.2) and (4.4) in (4.5), and eliminate **Q** from the resultant equations by means of (3.11). If we eliminate η among those resultant equations, we obtain

$$A_4{}^2\mathbf{P} = (\mathbf{A} \cdot \mathbf{P})\mathbf{A} \tag{D1}$$

after ignoring those terms containing μ^2. This relation is obviously satisfied, if

$$\mathbf{A} \parallel \mathbf{P} \tag{D2}$$

$$A^2 = A_4{}^2 \tag{D3}$$

Relations (D2) and (D3) are shown to exist as follows: Suppose that $\mathbf{A}'$ and A_4' are given; then, by a gauge transformation, we get

$$\mathbf{A} = \mathbf{A}' - \text{grad}\,\chi \tag{D4}$$

$$A_4 = A_4' - \dot{\chi}/c \tag{D5}$$

We may satisfy relation (D2) by choosing grad χ properly, and we may satisfy relation (D3) by choosing $\dot{\chi}$ properly. Conversely, relations (4.2), (4.4), (3.11) and (D1) lead to (4.5).

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